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A M S T E R D A M

STATISTISCHE AFDELING

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Kansverdelingen van quotiënten van stochastische grootheden

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Gevraagd wordt in het algemeen iets te zeggen over de verdeling van

$$\underline{z} = \underline{y}/\underline{x},$$

indien de simultane verdeling van  $\underline{x}$  en  $\underline{y}$  bekend is.

A. Indien verwachting en variantie van  $\underline{x}$  en  $\underline{y}$  worden voorgesteld door  $\mu_x, \sigma_x^2$ , resp.  $\mu_y, \sigma_y^2$  en de correlatiecoëfficiënt van  $\underline{x}$  en  $\underline{y}$  door  $\rho$ , kunnen de volgende benaderende uitdrukkingen voor verwachting en variantie van  $\underline{z}$  worden gevonden:

$$E \underline{z} = E \underline{y}/\underline{x} \approx \frac{\mu_y}{\mu_x} \left\{ 1 + \left( \frac{\sigma_x}{\mu_x} \right)^2 - \rho \cdot \frac{\sigma_x}{\mu_x} \cdot \frac{\sigma_y}{\mu_y} \right\}$$

$$\text{var } \underline{z} = \text{var } \underline{y}/\underline{x} \approx \left( \frac{\mu_y}{\mu_x} \right)^2 \left\{ \left( \frac{\sigma_x}{\mu_x} \right)^2 - 2\rho \frac{\sigma_x}{\mu_x} \cdot \frac{\sigma_y}{\mu_y} + \left( \frac{\sigma_y}{\mu_y} \right)^2 \right\}.$$

Deze uitdrukkingen vormen een bijzonder geval van het ondermeer door WALLIN [1] beschreven resultaat. Om de benadering te kunnen gebruiken moet evenwel aan de volgende voorwaarden zijn voldaan:

1°. Verwachting en variantie van  $\underline{z}$  moeten eindig zijn.

Hiervoor is in de eerste plaats noodzakelijk, dat de verdelingsdichtheid van  $\underline{x}$  in  $x=0$  nul is.

2°. De variatie-coëfficiënt  $\frac{\sigma_x}{\mu_x}$  van  $\underline{x}$  moet klein zijn.

B. De verdeling van het quotiënt van twee normaal verdeelde stochastische variabelen is uitvoerig onderzocht door GEARY [2] en FIELLER [3]. Voor kleine waarden van  $\frac{\sigma_x}{\mu_x}$  (bijv.  $\leq 1/3$ ) is

$$\frac{\mu_x \underline{z} - \mu_y}{\sqrt{\sigma_y^2 - 2\rho \cdot \sigma_x \sigma_y \cdot \underline{z} + \sigma_x^2 \underline{z}^2}}$$

bij benadering normaal verdeeld met verwachting 0 en variantie 1.

C. De verdeling van het quotiënt van een homogeen resp. driehoekig verdeelde variabele en een daarmee onafhankelijke, normaal verdeelde variabele is onderzocht door BROADBENT [4]. De door hem berekende tabellen zijn hierbij gevoegd.

- [1] J.F.E. WALLIN: Mean and variance of a function of two random variables.  
Statistica Neerlandica 13 (1959) p.31.
- [2] R.C. GEARY: The frequency distribution of the quotient of two normal variates.  
Journal of the Royal Statistical Society 93 (1930) p.442.
- [3] E.C. I The distribution of the index in a normal bivariate population.  
Biometrika 24 (1932) p.428.
- [4] S.R. BROADBENT: The quotient of a rectangular or triangular and a general variate.  
Biometrika 41 (1954) p.330.

The quotient of a rectangular or triangular and a general variate (Biometrika 41 (1954) p.336)

Table 1. Percentage points of the quotient of a rectangular and an independent normal variate. If  $y$  is rectangular with range  $(1-\alpha, 1+\alpha)$ , and  $x$  is normal with mean 1 and variance  $\beta^2$ , the table gives percentage points of  $(y/x)$

1% Points

| $100\beta \backslash 100\alpha$ | 0     | 2     | 4     | 6     | 8     | 10    |
|---------------------------------|-------|-------|-------|-------|-------|-------|
| 0                               | 1.000 | 0.980 | 0.961 | 0.941 | 0.922 | 0.902 |
| 1                               | .977  | .967  | .951  | .933  | .915  | .896  |
| 2                               | .956  | .949  | .935  | .919  | .903  | .885  |
| 3                               | .935  | .930  | .919  | .904  | .889  | .873  |
| 4                               | .915  | .911  | .902  | .889  | .875  | .859  |
| 5                               | .896  | .893  | .885  | .873  | .860  | .846  |

5% Points

| $100\beta \backslash 100\alpha$ | 0     | 2    | 4    | 6    | 8    | 10   |
|---------------------------------|-------|------|------|------|------|------|
| 0                               | 1.000 | .982 | .964 | .946 | .928 | .910 |
| 1                               | .984  | .975 | .960 | .944 | .927 | .909 |
| 2                               | .968  | .963 | .951 | .937 | .921 | .905 |
| 3                               | .953  | .950 | .940 | .928 | .914 | .898 |
| 4                               | .938  | .936 | .928 | .917 | .905 | .891 |
| 5                               | .924  | .922 | .916 | .906 | .895 | .882 |

95% Points

| $100\beta \backslash 100\alpha$ | 0     | 2     | 4     | 6     | 8     | 10    |
|---------------------------------|-------|-------|-------|-------|-------|-------|
| 0                               | 1.000 | 1.018 | 1.036 | 1.054 | 1.072 | 1.090 |
| 1                               | 1.017 | 1.025 | 1.040 | 1.057 | 1.074 | 1.091 |
| 2                               | 1.034 | 1.039 | 1.051 | 1.066 | 1.081 | 1.098 |
| 3                               | 1.052 | 1.055 | 1.065 | 1.078 | 1.092 | 1.107 |
| 4                               | 1.070 | 1.073 | 1.080 | 1.091 | 1.105 | 1.119 |
| 5                               | 1.090 | 1.092 | 1.098 | 1.108 | 1.119 | 1.133 |

99% Points

| $100\beta \backslash 100\alpha$ | 0     | 2     | 4     | 6     | 8     | 10    |
|---------------------------------|-------|-------|-------|-------|-------|-------|
| 0                               | 1.000 | 1.020 | 1.039 | 1.059 | 1.078 | 1.098 |
| 1                               | 1.024 | 1.034 | 1.051 | 1.069 | 1.087 | 1.106 |
| 2                               | 1.049 | 1.055 | 1.070 | 1.086 | 1.104 | 1.122 |
| 3                               | 1.075 | 1.080 | 1.092 | 1.107 | 1.123 | 1.140 |
| 4                               | 1.103 | 1.106 | 1.116 | 1.130 | 1.145 | 1.162 |
| 5                               | 1.132 | 1.135 | 1.143 | 1.155 | 1.169 | 1.185 |

Table 2. Percentage points of the quotient of a triangular and an independent normal variate. If  $y$  is triangular with range  $(1-2\alpha, 1+2\alpha)$ , and  $x$  is normal with mean 1 and variance  $\beta^2$ , the table gives percentage points of  $(y/x)$

1% Points

| $100\beta \backslash 100\alpha$ | 0     | 1     | 2     | 3     | 4     | 5     |
|---------------------------------|-------|-------|-------|-------|-------|-------|
| 0                               | 1.000 | 0.983 | 0.966 | 0.948 | 0.931 | 0.914 |
| 1                               | .977  | .971  | .958  | .943  | .927  | .911  |
| 2                               | .956  | .952  | .943  | .931  | .917  | .902  |
| 3                               | .935  | .932  | .925  | .916  | .904  | .891  |
| 4                               | .915  | .913  | .908  | .899  | .889  | .878  |
| 5                               | .896  | .894  | .890  | .883  | .874  | .864  |

5% Points

| $100\beta \backslash 100\alpha$ | 0     | 1     | 2     | 3     | 4     | 5     |
|---------------------------------|-------|-------|-------|-------|-------|-------|
| 0                               | 1.000 | 0.986 | 0.973 | 0.959 | 0.945 | 0.932 |
| 1                               | .984  | .979  | .969  | .956  | .943  | .930  |
| 2                               | .968  | .965  | .959  | .949  | .938  | .926  |
| 3                               | .953  | .951  | .946  | .939  | .929  | .919  |
| 4                               | .938  | .937  | .933  | .927  | .919  | .910  |
| 5                               | .924  | .922  | .919  | .915  | .908  | .900  |

95% Points

| $100\beta \backslash 100\alpha$ | 0     | 1     | 2     | 3     | 4     | 5     |
|---------------------------------|-------|-------|-------|-------|-------|-------|
| 0                               | 1.000 | 1.014 | 1.027 | 1.041 | 1.055 | 1.068 |
| 1                               | 1.017 | 1.021 | 1.032 | 1.044 | 1.057 | 1.070 |
| 2                               | 1.034 | 1.037 | 1.043 | 1.053 | 1.064 | 1.076 |
| 3                               | 1.052 | 1.054 | 1.059 | 1.066 | 1.076 | 1.086 |
| 4                               | 1.070 | 1.072 | 1.075 | 1.082 | 1.090 | 1.099 |
| 5                               | 1.090 | 1.091 | 1.094 | 1.099 | 1.106 | 1.114 |

99% Points

| $100\beta \backslash 100\alpha$ | 0     | 1     | 2     | 3     | 4     | 5     |
|---------------------------------|-------|-------|-------|-------|-------|-------|
| 0                               | 1.000 | 1.017 | 1.034 | 1.052 | 1.069 | 1.086 |
| 1                               | 1.024 | 1.030 | 1.043 | 1.058 | 1.074 | 1.090 |
| 2                               | 1.049 | 1.052 | 1.062 | 1.074 | 1.088 | 1.103 |
| 3                               | 1.075 | 1.078 | 1.085 | 1.094 | 1.106 | 1.120 |
| 4                               | 1.103 | 1.105 | 1.110 | 1.118 | 1.129 | 1.140 |
| 5                               | 1.132 | 1.134 | 1.138 | 1.145 | 1.154 | 1.164 |